Exam Numerieke Wiskunde 1
June 22, 2012

Use of a simple calculator is allowed. All answers need to be motivated.
In front of the exercises you find its weight. In fact it gives the number of tenths which can be gained in the final mark. In total 6 points can be scored with this exam.

1. In the table below the results of a computation with two different fixed point methods $x_{n+1} = g(x_n)$ and $y_{n+1} = h(y_n)$ are given for the same problem.

n	x_n	y_n
0	1.0	1.0
1	1.06341161	1.06240557
2	1.06146661	1.06154993
3	1.06155345	1.06154977
4	1.06154961	1.06154977
5	1.06154978	1.06154977

   (a) $\boxed{3}$ Give an estimate of $g'(p)$ and $h'(p)$ at $n = 2, 3$ and 4 where p is the fixed point.

   (b) $\boxed{3}$ If possible give an Aitken error estimate for x_3 and y_3. Indicate why or why not such an estimate can be made.

   (c) $\boxed{3}$ How can the error estimate(s) of part (b) be used to improve x_3 and/or y_3 in the case(s) where it can be applied? Compute this (these) improvement(s).

2. Suppose $f_1 = x_1^2 + 4x_2^2 - 4, \quad f_2 = x_1 + x_2 - \ln(1 + x_2 - x_1)$. We want to solve $f_1(x_1, x_2) = 0, \quad f_2(x_1, x_2) = 0$, which is written in vector form as $\mathbf{f}(\mathbf{x}) = \mathbf{0}$. We want to solve it with a fixed point method $\mathbf{x_{n+1}} = \mathbf{g}(\mathbf{x_n})$ where $\mathbf{g}(\mathbf{x}) = \mathbf{x} + A\mathbf{f}(\mathbf{x})$.

   (a) $\boxed{3}$ Give the Jacobian matrix of $\mathbf{f}$.

   (b) $\boxed{2}$ Why is $(0, 1)$ a reasonable guess of the zero?

   (c) $\boxed{4}$ Give the equation from which the best matrix A, based on the guess in the previous part, must be solved.

   (d) $\boxed{3}$ The Jacobian of $\mathbf{g}$ at the fixed point using the A of the previous part is

$$\begin{bmatrix} -0.0518422 & 0.099914 \\ -0.113687 & -0.0305281 \end{bmatrix}$$

   Why is this matrix relevant for the study of the convergence of the fixed point method? Will the method converge in the neighborhood of the fixed point and why?

3. Given the function $f(x) = \sin(x)$ and the two interpolation points $x = 0$ and $x = 1$.

   (a) $\boxed{2}$ Give the general form of of the interpolation polynomial in Newton divided differences on two interpolation points x_0 and x_1. Also give the error term. Compute the occuring Newton divided differences on the interpolation points for the function above.

   (b) $\boxed{2}$ Give the interpolating polynomial based on the interpolation points $x = 0$ en $x = 1$. Use this to approximate the value of $f(x)$ at $x = 1/2$, given that $\sin(1) = 0.841471$.

   (c) $\boxed{2}$ Give the maximum interpolation error on the interval $[0, 1]$ based on the interpolation error term.

   (d) $\boxed{2}$ Given $\sin(1/2) = 0.479426$, compare the actual error to the maximum interpolation error found in (c). Is the actual error consistent with the result found in (c)?

Continue on other side!

4. (a) [3] Simpson's rule for integration on the interval $[a, b]$ is given by

$$I_S = \frac{b-a}{6}\{\ f(a) + 4f(\frac{a+b}{2}) + f(b)\ \}$$

Give the interpolation polynomial on which this rule is based, expressed in Lagrangian basis functions. Also indicate how from this Simpson's rule is derived.

(b) [3] The trapezium rule is based on two interpolation points and is exact for linear polynomials. One would expect that Simpson's rule, which uses one extra interpolation point, is exact for quadratic polynomials. It is however exact for cubic polynomials. Show that this is the case and explain why this occurs.

(c) [3] Suppose we have a numerical method which approximates I by $I(h)$, where h is the mesh size and $I = I(h) + ch + O(h^2)$ for some nonzero c. Derive a combination of $I(h)$ and $I(2h)$ that approximates I and which has error $O(h^2)$.

5. (a) [3] Describe the power method to find the largest eigenvalue of a matrix. Also show that the convergence depends on $|\lambda_2/\lambda_1|$ where λ_1 is the biggest eigenvalue and λ_2 the one but biggest eigenvalue and not equal in magnitude to λ_1.

(b) [3] For which kind of matrices is the solution x of $Ax = b$ the same as the minimum of $J(x) = \frac{1}{2}(x, Ax) - (x, b)$ over x. Show this by computing the minimum of J.

(c) [2] Explain the idea of the gradient method and argue why in this method $J(x)$ is monotonically decreasing.

(d) [2] In (conjugate) gradient methods it is often beneficial to use preconditioning. Explain what preconditioning is and what its aim is.

6. Consider on [0, 1] for $u(x,t)$ the convection equation $\partial u/\partial t = -c\partial u/\partial x$, with $c = 5$, and initial condition $u(x, 0) = \sin(\pi x)$ and boundary conditions $u(0, t) = \sin^2(t)$. Let the grid in x direction be given by $x_i = i\Delta x$ where $\Delta x = 1/m$.

(a) [3] Show that $u_x(x_i, t) = \frac{u(x_i,t)-u(x_{i-1},t)}{\Delta x} + O(\Delta x)$.

(b) [3] Show that the system of ordinary differential equations (ODEs) that results from using the expression in (a) is of the form

$$\frac{d}{dt}\vec{u}(t) = -\frac{c}{\Delta x}(I - B)\vec{u}(t) + \vec{b}(t)$$

and give B and $\vec{b}(t)$.

(c) [3] Show that any eigenvalue of B will be at most one in magnitude. Sketch in the complex plane where the eigenvalues of $I - B$ are located.

(d) [3] Show that the numerical integration with the Forward/Explicit Euler method of the system of ODEs will be stable if $c\Delta t/\Delta x \leq 1$.

Total [60]